Discrete Mathematics 309 (2009) 6026–6034

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A construction of d^z-disjunct matrices by orthogonal space and discussion on their design parameters[☆]

Geng-sheng Zhang [a,*], Yu-qin Yang [a], Xiang-hui Zhao [b]

[a] Department of Mathematics, Hebei Normal University, Shijiazhuang, 050016, PR China

[b] Institute of Science, Hebei University of Science and Technology, Shijiazhuang, 050018, PR China

ARTICLE INFO

Article history:
Received 15 May 2008
Received in revised form 16 April 2009
Accepted 1 May 2009
Available online 24 May 2009

Keywords:
Orthogonal space
Group testing
d^z-disjunct
Design parameters
Principal part

ABSTRACT

In this paper, we construct a class of d^z-disjunct matrices with high degree of error-correction (detection) by geometry of orthogonal group over finite field of odd characteristic and discuss the disjunct properties of them. For a $t \times n$ d^z-disjunct matrix, it has four parameters, n, t, d, z, which we call the design parameters of the d^z-disjunct matrix. In order to facilitate the selection of a pooling design in line with our need, we have discussed how these design parameters change with their variables based on the construction.

1. Introduction

A group testing that indicates whether or not a positive data object is in a specific subset or pool of the data set can greatly facilitate the identification of all the positive data objects. This job can be done through a collection of tests, called a pooling design. Each test is a subset of clones, called a pool. A clone is positive if it contains a given probe, otherwise, it is negative. A pool is positive if and only if it contains at least one positive clone, otherwise it is negative.

Pooling designs have become the standard experimental tools in many areas of application. A group testing algorithm is non-adaptive if all tests could be specified without knowing the outcomes of other tests. Designing a good error-tolerance pooling design is the central problem in the area of non-adaptive group testing. A mathematical model of the non-adaptive group testing (NGT) design is a d-disjunct matrix. A $(0, 1)$-matrix is called d-disjunct if no union of any d columns covers another column. A $t \times n$ d-disjunct matrix corresponds precisely to a pooling design which can identify at most d positive items from n items with t tests. A d-disjunct matrix is d^z-disjunct if no column has at least z 1-entries covered by the union of any other d columns. Hence, a d-disjunct matrix is d^1-disjunct. A d^z-disjunct matrix can detect $z-1$ errors and correct $\lfloor \frac{z-1}{2} \rfloor$ errors (see [1,5]). If an extra round of confirmatory tests is allowed, then a d^z-disjunct matrix can indeed correct $z-1$ errors (see [3]). For a $t \times n$ d^z-disjunct matrix, it is clear that the larger n, $\frac{1}{t}$, d, z are, the better the pooling design is.

To date, there are a few relative papers addressing the problem of designing and analyzing error-tolerance pooling design (see [7,9,6,8,2,11,4]). Anthony J. Macula proposed a way of constructing a d-disjunct matrix which uses the containment relation in a structure [7]. In [9], Ngo and Du introduced a non-adaptive pooling design based on the containment relation of finite vector space which was later found to be highly error-tolerance by D'yachkov et al. in [2].

☆ Supported by NSF of Hebei Province (A2009000253), NSF of the Education Department of Hebei Province (2007127) and NSF of Hebei Normal University (L2004B04).

* Corresponding author.
E-mail addresses: gshzhang@hebtu.edu.cn, zjfwdl@163.com (G.-s. Zhang).

0012-365X/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2009.05.001

If we also take the type of the subspaces into our consideration rather than only the dimension of them as in [9,2], we believe that we can get some better conclusions about the pooling design constructed by orthogonal space. In this paper, we construct a d^z-disjunct matrix whose rows are indexed by $(r, 0, 0)$-subspaces and columns indexed by $(m, 0, 0)$-subspaces of orthogonal space and find that it has a high degree of error-correction (detection) as in [2]. In addition, we give the formulas of n, t, d and z, and analyze the changing trends of them with their variables.

2. Preliminary

In order to construct a d^z-disjunct matrix with orthogonal space over finite field of odd characteristic, we introduce several basic conceptions and notations of orthogonal geometry. For more results of the orthogonal geometry, the reader can refer to Chapter 6 of [10].

$\mathbb{F}_q$ is a finite field of odd characteristic with q elements, where q is a prime or a prime power. w is a fixed non-square element of $\mathbb{F}_q^*$. Assume

$$S_{2\nu,0}=\begin{pmatrix}0 & I^{(\nu)}\\ I^{(\nu)} & 0\end{pmatrix},\qquad S_{2\nu+1,1}=\begin{pmatrix}0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & 1\end{pmatrix},$$

$$S_{2\nu+1,w}=\begin{pmatrix}0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & w\end{pmatrix},\qquad S_{2\nu+2,w}=\begin{pmatrix}0 & I^{(\nu)} & & \\ I^{(\nu)} & 0 & & \\ & & 1 & \\ & & & -w\end{pmatrix}.$$

In order to cover these four cases, we introduce the notation $S_{2\nu+\delta,\Delta}$, where ν is its index and Δ denotes its definite part. The orthogonal group of degree $2\nu+\delta$ with respect to $S_{2\nu+\delta,\Delta}$ over $\mathbb{F}_q$ will be denoted by $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$, which consists of all $(2\nu+\delta)\times(2\nu+\delta)$ matrices T over $\mathbb{F}_q$ satisfying $TS_{2\nu+\delta,\Delta}T^{\mathrm{T}} = S_{2\nu+\delta,\Delta}$. The vector space $\mathbb{F}_q^{2\nu+\delta}$ together with the right multiplication action of $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$ is called the $(2\nu+\delta)$-dimensional orthogonal space over $\mathbb{F}_q$ with respect to $S_{2\nu+\delta,\Delta}$. Clearly, since $O_{2\nu+1,1}(\mathbb{F}_q)$ and $O_{2\nu+1,w}(\mathbb{F}_q)$ are isomorphic, actually only three types of orthogonal groups, $O_{2\nu,0}(\mathbb{F}_q)$, $O_{2\nu+1,1}(\mathbb{F}_q)$, $O_{2\nu+2,w}(\mathbb{F}_q)$, need to be considered.

Let A and B be two $n\times n$ matrices over $\mathbb{F}_q$. If there is an $n\times n$ non-singular matrix Q over $\mathbb{F}_q$ such that $QAQ^T=B$, we call that A is cogradient to B.

Let P be an m-dimensional vector subspace of $\mathbb{F}_q^{2\nu+\delta}$. We use the same letter to denote the matrix which represents the vector subspace P. It should be noted that the matrix P is an $m\times(2\nu+\delta)$ matrix of rank m. Subspace P is said to be of type $(m, 2s, s)$, if $PS_{2\nu+\delta,\Delta}P^{\mathrm{T}}$ is cogradient to the matrix

$$\begin{pmatrix}0 & I^{(s)} & \\ I^{(s)} & 0 & \\ & & 0^{(m-2s)}\end{pmatrix}.$$

The subspaces of type $(m, 0, 0)$ are called m-dimensional totally isotropic subspaces, denoted by $(m, 0, 0)$-subspace. Two vectors x and y of $\mathbb{F}_q^{2\nu+\delta}$ are said to be orthogonal with respect to $S_{2\nu+\delta,\Delta}$, if $xS_{2\nu+\delta,\Delta}y^{\mathrm{T}}=0$. We use the symbol $P^\perp$ to denote the set of vectors which are orthogonal to every vector of P, i.e., $P^\perp=\{y\in\mathbb{F}_q^{2\nu+\delta} \mid yS_{2\nu+\delta,\Delta}x^t=0, \text{ for all } x\in P\}$. Obviously, $P^\perp$ is a $(2\nu+\delta-m)$-dimensional subspace of $\mathbb{F}_q^{2\nu+\delta}$ and is called the dual subspace of P. A subspace P is totally isotropic if and only if $P\subseteq P^\perp$. Denote the dual subspace of $(m, 0, 0)$-subspace with respect to $S_{2\nu+\delta,\Delta}$ in $\mathbb{F}_q^{2\nu+\delta}$ by $(m, 0, 0)^\perp$.

Denote by $\mathcal{M}(m, 0, 0; 2\nu+\delta, \Delta)$ the set of subspaces of type $(m, 0, 0)$ contained in $\mathbb{F}_q^{2\nu+\delta}$. Let $N(m, 0, 0; 2\nu+\delta, \Delta) = |\mathcal{M}(m, 0, 0; 2\nu+\delta, \Delta)|$.

Let P be a fixed subspace of type $(m, 0, 0)$ of $\mathbb{F}_q^{2\nu+\delta}$. Denote by $\mathcal{M}(m_1, 0, 0; m, 0, 0; 2\nu+\delta, \Delta)$ the set of subspaces of type $(m_1, 0, 0)$ contained in P. Let $N(m_1, 0, 0; m, 0, 0; 2\nu+\delta, \Delta) = |\mathcal{M}(m_1, 0, 0; m, 0, 0; 2\nu+\delta, \Delta)|$. Denote by $\mathcal{M}'(m, 0, 0; m_2, 0, 0; 2\nu+\delta, \Delta)$ the set of subspaces of type $(m_2, 0, 0)$ containing P. Let $N'(m, 0, 0; m_2, 0, 0; 2\nu+\delta, \Delta) = |\mathcal{M}'(m, 0, 0; m_2, 0, 0; 2\nu+\delta, \Delta)|$.

Additionally, three Anzahl theorems which are used in this paper are as follows. According to Corollary 6.23, Theorem 6.33 and 6.43 in [10] respectively, we can obtain them directly.

Lemma 2.1 (*[10]*). *If $m\le\nu$, then*

$$N(m, 0, 0; 2\nu+\delta, \Delta)=\frac{\prod_{i=\nu-m+1}^{\nu}(q^i-1)(q^{i+\delta-1}+1)}{\prod_{i=1}^{m}(q^i-1)}.$$

Lemma 2.2 ([10]). *If* $m \le \nu$, *then*

$$N(m_1, 0, 0; m, 0, 0; 2\nu+\delta, \Delta) = \frac{\prod_{i=m-m_1+1}^{m}(q^i-1)}{\prod_{i=1}^{m_1}(q^i-1)}.$$

Lemma 2.3 ([10]). *If* $m \le \nu$, *then*

$$N'(m, 0, 0; m_2, 0, 0; 2\nu+\delta, \Delta) = \frac{\prod_{i=1}^{\nu-m}(q^i-1)(q^{i-1}+1)}{\prod_{i=1}^{\nu-m_2}(q^i-1)\prod_{i=0}^{\nu-m_2-1}(q^i+1)\prod_{i=1}^{m_2-m}(q^i-1)}.$$

We adopt the convention that $\prod_{i\in M} f(i) = 1$, when M denotes the empty set.

3. The construction

Definition 3.1. Select integers $0 \le m_0 < r < m \le \nu$. Assume P_0 is a fixed $(m_0, 0, 0)$-subspace of $\mathbb{F}_q^{2\nu+\delta}$. Let M be the $(0, 1)$-matrix, where the columns (rows) are labeled by $(m, 0, 0)$-subspaces ($(r, 0, 0)$-subspaces), which are contained in $P_0^{\perp}$ and contain P_0. $m_{ij} = 1$ if and only if the label of row i is contained in the label of column j.

We can show that M is a d^z-disjunct matrix with certain constant weights.

Proposition 3.2. *M is an $N(r-m_0, 0, 0; 2(\nu-m_0)+\delta, \Delta) \times N(m-m_0, 0, 0; 2(\nu-m_0)+\delta, \Delta)$ matrix, whose constant row (column) weight is $N'(r-m_0, 0, 0; m-m_0, 0, 0; 2(\nu-m_0)+\delta, \Delta)$ $(N(r-m_0, 0, 0; m-m_0, 0, 0; 2(\nu-m_0)+\delta, \Delta))$.*

Proof. By the transitivity of $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$ on the set of subspaces of the same type (see [10]), we can assume that P_0 and $P_0^{\perp}$ have matrix representations of the forms

$$P_0 = \begin{array}{c} \begin{pmatrix} I^{(m_0)} & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \begin{array}{ccccc} m_0 & \nu-m_0 & m_0 & \nu-m_0 & \delta \end{array} \end{array}$$

$$P_0^{\perp} = \begin{array}{c} \begin{pmatrix} I^{(m_0)} & 0 & 0 & 0 & 0 \\ 0 & I^{(\nu-m_0)} & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(\nu-m_0)} & 0 \\ 0 & 0 & 0 & 0 & I^{\delta} \end{pmatrix}. \\ \begin{array}{ccccc} m_0 & \nu-m_0 & m_0 & \nu-m_0 & \delta \end{array} \end{array}$$

Let R be a $(r, 0, 0)$-subspace contained in $P_0^{\perp}$ and containing P_0. Then

$$R = \begin{array}{cc} \begin{pmatrix} I^{(m_0)} & 0 & 0 & 0 & 0 \\ 0 & R_1 & R_2 & R_3 & R_4 \end{pmatrix} & \begin{array}{c} m_0 \\ r-m_0 \end{array} \\ \begin{array}{ccccc} m_0 & \nu-m_0 & m_0 & \nu-m_0 & \delta \end{array} & \end{array}.$$

Since R is a $(r, 0, 0)$-subspace, i.e., the rank of $RS_{2\nu+\delta,\Delta}R^t$ is 0, we have $R_2 = 0$ and further $\begin{pmatrix} R_1 & R_3 & R_4 \end{pmatrix}$ represents a $(r-m_0, 0, 0)$-subspace in $\mathbb{F}_q^{2(\nu-m_0)+\delta}$. Hence the number of rows in M is $N(r-m_0, 0, 0; 2(\nu-m_0)+\delta, \Delta)$ by Lemma 2.1.

Let Q be a $(m, 0, 0)$-subspace contained in $P_0^{\perp}$ and containing P_0. Then

$$Q = \begin{array}{cc} \begin{pmatrix} I^{(m_0)} & 0 & 0 & 0 & 0 \\ 0 & Q_1 & Q_2 & Q_3 & Q_4 \end{pmatrix} & \begin{array}{c} m_0 \\ m-m_0 \end{array} \\ \begin{array}{ccccc} m_0 & \nu-m_0 & m_0 & \nu-m_0 & \delta \end{array} & \end{array}.$$

Similarly, $Q_2 = 0$ and $\begin{pmatrix} Q_1 & Q_3 & Q_4 \end{pmatrix}$ represents a $(m-m_0, 0, 0)$-subspace in $\mathbb{F}_q^{2(\nu-m_0)+\delta}$ and the number of columns in M is $N(m-m_0, 0, 0; 2(\nu-m_0)+\delta, \Delta)$.

The row weight is the number of $(m-m_0, 0, 0)$-subspaces containing a fixed $(r-m_0, 0, 0)$-subspace in $\mathbb{F}_q^{2(\nu-m_0)+\delta}$. Hence it is $N'(r-m_0, 0, 0; m-m_0, 0, 0; 2(\nu-m_0)+\delta, \Delta)$ by Lemma 2.3.

Similarly, the column weight is the number of $(r-m_0, 0, 0)$-subspaces contained in a fixed $(m-m_0, 0, 0)$-subspace in $\mathbb{F}_q^{2(\nu-m_0)+\delta}$. Hence it is $N(r-m_0, 0, 0; m-m_0, 0, 0; 2(\nu-m_0)+\delta, \Delta)$ by Lemma 2.2. ∎

Theorem 3.3. *M is d-disjunct, where $d = r - m_0$.*

Proof. Let $C_0, C_1, \ldots, C_{r-m_0}$ be $r - m_0 + 1$ distinct columns of M, i.e., $(m, 0, 0)$-subspaces contained in $P_0^{\perp}$ and containing P_0. Suppose column C_0 is designated. Since $C_0, C_1, \ldots, C_{r-m_0}$ are different subspaces which all contain P_0, there must be $(1, 0, 0)$-subspaces A_i not contained in P_0 satisfying $A_i \subseteq C_0 \setminus C_i$ $(1 \le i \le r - m_0)$. Set $A = \bigoplus_{i=1}^{r-m_0} A_i \bigoplus P_0$. Then A is a $(y, 0, 0)$-subspace, where $y \le r$. Take a $(r - y, 0, 0)$-subspace A^* from the complementary space of A in C_0. Then $A \bigoplus A^*$ is a $(r, 0, 0)$-subspace contained in $P_0^{\perp}$ and containing P_0. Moreover, $A \bigoplus A^* \subseteq C_0 \setminus C_i (1 \le i \le r - m_0)$. Therefore, at row $A \bigoplus A^*$, column C_0 has 1-entry and columns $C_1, \ldots, C_{r-m_0}$ have 0-entry. Hence M is d- disjunct, where $d = r - m_0$. ■

Theorem 3.4. *Suppose $m - r \ge 2$ and set $b_1 = \frac{q(q^{m-m_0-1}-1)}{q^{m-r}-1}$. Then M is d^z-disjunct when $1 \le d \le b_1$, where*

$$\begin{aligned} z &= N(r - m_0, 0, 0; m - m_0, 0, 0; 2(\nu - m_0) + \delta, \Delta) \\ &\quad - dN(r - m_0, 0, 0; m - m_0 - 1, 0, 0; 2(\nu - m_0) + \delta, \Delta) \\ &\quad + (d - 1)N(r - m_0, 0, 0; m - m_0 - 2, 0, 0; 2(\nu - m_0) + \delta, \Delta). \end{aligned}$$

Proof. Let $C_0, C_1, \ldots, C_d$ be $d+1$ distinct columns of M, i.e., $(m, 0, 0)$-subspaces contained in $P_0^{\perp}$ and containing P_0. Suppose column C_0 is designated. There are $N(r - m_0, 0, 0; m - m_0, 0, 0; 2(\nu - m_0) + \delta, \Delta)$ $(r, 0, 0)$-subspaces containing P_0 in C_0. Let $|C_0 \setminus \bigcup_{i=1}^{d} C_i|$ be the number of $(r, 0, 0)$-subspaces containing P_0 and contained in C_0 but not contained in C_i $(1 \le i \le d)$. Noticing that $|C_0 \setminus \bigcup_{i=1}^{d} C_i|$= $|C_0 \setminus \bigcup_{i=1}^{d}(C_0 \bigcap C_i)|$, to obtain the minimum of $|C_0 \setminus \bigcup_{i=1}^{d} C_i|$, we may assume that each C_i intersects C_0 at a $(m - 1, 0, 0)$-subspace. Then each C_i covers $N(r - m_0, 0, 0; m - m_0 - 1, 0, 0; 2(\nu - m_0) + \delta, \Delta)(r, 0, 0)$-subspaces containing P_0 of C_0. However, by our hypothesis each pair of C_i and C_j overlaps at an $(m - 2, 0, 0)$-subspace containing P_0. Therefore C_1 covers the full $N(r - m_0, 0, 0; m - m_0 - 1, 0, 0; 2(\nu - m_0) + \delta, \Delta)$ $(r, 0, 0)$-subspaces, while each of $C_2, \ldots, C_d$ can cover the maximum of $N(r - m_0, 0, 0; m - m_0 - 1, 0, 0; 2(\nu - m_0) + \delta, \Delta) - N(r - m_0, 0, 0; m - m_0 - 2, 0, 0; 2(\nu - m_0) + \delta, \Delta)$ $(r, 0, 0)$-subspaces containing P_0 and not covered by C_1. Consequently the number of $(r, 0, 0)$-subspaces containing P_0 and contained in C_0 but not contained in C_i $(1 \le i \le d)$ is at least

$$\begin{aligned} z &= N(r - m_0, 0, 0; m - m_0, 0, 0; 2(\nu - m_0) + \delta, \Delta) - dN(r - m_0, 0, 0; m - m_0 - 1, 0, 0; 2(\nu - m_0) + \delta, \Delta) \\ &\quad + (d - 1)N(r - m_0, 0, 0; m - m_0 - 2, 0, 0; 2(\nu - m_0) + \delta, \Delta). \end{aligned}$$

For a d^z-disjunct matrix, z must be positive, which implies

$$d < \frac{q(q^{m-m_0-1} - 1)}{q^{m-r} - 1} + 1.$$

Set $b_1 = \frac{q(q^{m-m_0-1}-1)}{q^{m-r}-1}$. Then $1 \le d \le b_1$. ■

Let C_0 be a column of M, namely a $(m, 0, 0)$-subspace contained in $P_0^{\perp}$ and containing P_0, and E be a fixed $(m - 2, 0, 0)$-subspace containing P_0 and contained in C_0. Let F be a $(m - 1, 0, 0)$-subspace containing E and contained in C_0. By the transitivity of $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$ on the set of subspaces of the same type, we can assume that

$$F = \begin{array}{c} \left(\begin{array}{cccccc} I^{(m_0)} & 0 & 0 & 0 & 0 & 0 \\ 0 & I^{(m-m_0-2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & F_1 & F_2 & F_3 & F_4 \end{array} \right) \begin{array}{l} m_0 \\ m - m_0 - 2 \\ 1 \end{array} . \\ \begin{array}{cccccc} m_0 & m - m_0 - 2 & \nu - m + 2 & m - 2 & \nu - m + 2 & \delta \end{array} \end{array}$$

Since F is a $(m - 1, 0, 0)$-subspace, i.e., the rank of $FS_{2\nu+\delta,\Delta}F^{\mathrm{T}}$ is 0, we have $F_2 = 0$ and $\begin{pmatrix} F_1 & F_3 & F_4 \end{pmatrix}$ represents a $(1, 0, 0)$-subspace in $\mathbb{F}_q^{2(\nu-m+2)+\delta}$. Hence the number of F is $N(1, 0, 0; 2(\nu - m + 2) + \delta, \Delta)$.

$$N(1, 0, 0; 2(\nu - m + 2) + \delta, \Delta) = \frac{(q^{\nu-m+2} - 1)(q^{\nu-m+1+\delta} + 1)}{q - 1}.$$

Set $b_2 = \frac{(q^{\nu-m+2}-1)(q^{\nu-m+1+\delta}+1)}{q-1}$.

Theorem 3.5. *Suppose $m - r \ge 2$ and $1 \le d \le b$, where $b = \min\{b_1, b_2\}$. Then M is not d^{z+1}-disjunct, where b_1 and z are as in Theorem 3.4 and b_2 is as above.*

Proof. We will show that the minimum of $|C_0 \setminus C_1 \bigcup \ldots \bigcup C_d|$ in the proof of Theorem 3.4 can be obtained. Since $1 \le d \le b$, we can choose d distinct $(m - 1, 0, 0)$-subspaces between E and C_0, named F_i $(1 \le i \le d)$ and for each F_i, we can choose a $(m, 0, 0)$-subspace C_i such that $C_0 \bigcap C_i = F_i$. Hence each pair of C_i and C_j overlaps at the same $(m-2, 0, 0)$-subspace E. ■

If r takes its minimum m_0+1, we have the following corollary.

Corollary 3.6. *Suppose $r=m_0+1$, $m-r\geq 2$ and $1\leq d\leq q$. Then M is d^z-disjunct, but not d^{z+1}-disjunct, where*

$$z=q^{m-m_0-2}(q-1)(q-d+1).$$

Proof. Setting $r=m_0+1$ in the z formula of Theorem 3.4, we obtain

$$z=q^{m-m_0-2}(q-1)(q-d+1).$$

The second statement follows directly from Theorem 3.5. ■

Moreover, if r, m, ν all take their minimums, then z will be in a neater form.

Corollary 3.7. *Suppose $r=m_0+1$, $m=r+2=m_0+3=\nu$ and $1\leq d\leq q$. Then M is d^z-disjunct, but not d^{z+1}-disjunct, where*

$$z=q(q-1)(q-d+1).$$

Now we give an example based on our construction more specifically.

Example 3.8. Choose $q=3$, $m_0=1$, $r=2$, $m=4$, $\delta=1$ and $\nu=4$. Then M is 3^6-disjunct with 364 rows and 1120 columns. This means that our method to identify at most 3 positives from 1120 items with 364 tests and 5 errors can be detected and corrected.

4. The discussion on design parameters

We generally use the following four parameters, n, t, d, z, which we call the design parameters of a d^z-disjunct matrix, to evaluate whether a pooling design is judicious or not. According to the introduction in Section 1, n denotes the number of items the design could identify in an experiment, t the number of tests required by the design, d the greatest number of positive items allowed by the design and z the capability of error-tolerance and error-correction of the design. Therefor for a d^z-disjunct matrix, we expect that the values of n, d, z are as large as they could and t as small as it could. However, generally speaking, the design parameters all rely on more than one variables and these variables may appear in different formulas of those design parameters at the same time, which lead to the design parameters influencing each other. So it is very complex to discuss them comprehensively. In order to facilitate the selection of a pooling design in line with our need, we discuss how these design parameters change with their variables based on the construction in this paper.

Firstly we study how the number of columns, i.e., $n(\nu, m, m_0)$, is affected by its variables. The similar deduction can also apply to the discussion of the number of rows.

Since

$$n(\nu,m,m_0)=N(m-m_0,0,0;2(\nu-m_0)+\delta,\Delta)=\frac{\prod_{i=\nu-m+1}^{\nu-m_0}(q^i-1)(q^{i+\delta-1}+1)}{\prod_{i=1}^{m-m_0}(q^i-1)},$$

it is easy to get the conclusion that $n(\nu, m, m_0)$ increases as ν increases but decreases when m_0 increases. Additionally, the following theorem tells us how to choose m so that the number of items the design could identify in an experiment reaches its maximum.

Theorem 4.1. *For fixed ν, m_0, when $0\leq m_0<m\leq\nu$, the sequence $N(m-m_0,0,0;2(\nu-m_0)+\delta,\Delta)$ is unimodal and gets its peak at $m=\lceil\frac{2\nu+m_0+\delta-2}{3}\rceil$ or $m=\lfloor\frac{2\nu+m_0+\delta+1}{3}\rfloor$.*

Proof. Assume $m_0<m_1<m_2\leq\nu$. By Lemma 2.1, we have

$$\begin{aligned}\frac{N(m_2-m_0,0,0;2(\nu-m_0)+\delta,\Delta)}{N(m_1-m_0,0,0;2(\nu-m_0)+\delta,\Delta)}&=\frac{\prod_{i=\nu-m_2+1}^{\nu-m_0}(q^i-1)(q^{i+\delta-1}+1)}{\prod_{i=1}^{m_2-m_0}(q^i-1)}\Bigg/\frac{\prod_{i=\nu-m_1+1}^{\nu-m_0}(q^i-1)(q^{i+\delta-1}+1)}{\prod_{i=1}^{m_1-m_0}(q^i-1)}\\&=\frac{\prod_{i=\nu-m_2+1}^{\nu-m_1}(q^i-1)(q^{i+\delta-1}+1)}{\prod_{i=m_1-m_0+1}^{m_2-m_0}(q^i-1)}\end{aligned}$$

$$= \frac{(q^{\nu-m_1}-1)(q^{\nu-m_1+\delta-1}+1)}{(q^{m_1-m_0+1}-1)} \cdot \frac{(q^{\nu-m_1-1}-1)(q^{\nu-m_1+\delta-2}+1)}{(q^{m_1-m_0+2}-1)} \cdots \frac{(q^{\nu-m_2+1}-1)(q^{\nu-m_2+\delta}+1)}{(q^{m_2-m_0}-1)}.$$

We know that

$$\frac{(q^{\nu-m_1}-1)(q^{\nu-m_1+\delta-1}+1)}{(q^{m_1-m_0+1}-1)} > \frac{(q^{\nu-m_1-1}-1)(q^{\nu-m_1+\delta-2}+1)}{(q^{m_1-m_0+2}-1)} > \cdots > \frac{(q^{\nu-m_2+1}-1)(q^{\nu-m_2+\delta}+1)}{(q^{m_2-m_0}-1)}.$$

So we have that if

$$\frac{(q^{\nu-m_1}-1)(q^{\nu-m_1+\delta-1}+1)}{(q^{m_1-m_0+1}-1)} < 1,$$

i.e., $\nu \geq m_2 > m_1 \geq \lceil \frac{2\nu+m_0+\delta-2}{3} \rceil$, then

$$N(m_2-m_0,0,0;2(\nu-m_0)+\delta,\Delta) < N(m_1-m_0,0,0;2(\nu-m_0)+\delta,\Delta);$$

if

$$\frac{(q^{\nu-m_2+1}-1)(q^{\nu-m_2+\delta}+1)}{(q^{m_2-m_0}-1)} > 1,$$

i.e., $m_0 < m_1 < m_2 \leq \lfloor \frac{2\nu+m_0+\delta+1}{3} \rfloor$, then

$$N(m_2-m_0,0,0;2(\nu-m_0)+\delta,\Delta) > N(m_1-m_0,0,0;2(\nu-m_0)+\delta,\Delta).$$

Noting

$$\left| \frac{2\nu+m_0+\delta-2}{3} - \frac{2\nu+m_0+\delta+1}{3} \right| = 1,$$

so the sequence $N(m-m_0,0,0;2(\nu-m_0)+\delta,\Delta)$ is unimodal and reaches its peak at $m = \lceil \frac{2\nu+m_0+\delta-2}{3} \rceil$ or $m = \lfloor \frac{2\nu+m_0+\delta+1}{3} \rfloor$. ■

In fact, so long as $\frac{2\nu+m_0+\delta-2}{3}$ is not an integer, then $\lceil \frac{2\nu+m_0+\delta-2}{3} \rceil = \lfloor \frac{2\nu+m_0+\delta+1}{3} \rfloor$.

Let $f(q)$ and $g(q)$ be monic polynomials with degrees s and t respectively, where $s \geq t$. We call q^{s-t} is the principal part of $\frac{f(q)}{g(q)}$, for $\lim_{q\to\infty} \frac{f(q)}{g(q)q^{s-t}} = 1$.

Considering the principal part of $n(\nu, m, m_0)$, $q^{(4\nu-3m-m_0+2\delta-1)(\frac{m-m_0}{2})}$, we conclude that $q^{(4\nu-3m-m_0+2\delta-1)(\frac{m-m_0}{2})}$ increases when ν increases but decreases as m_0 increases. If $m = \langle \frac{2\nu+m_0+\delta-1/2}{3} \rangle$ ($\langle x \rangle$ represents the integer closest to x), then $q^{(4\nu-3m-m_0+2\delta-1)(\frac{m-m_0}{2})}$ reaches the maximum, which is consistent with the results of our theorem. According to this observation, we can say that the principal part of a design parameter not only makes our discussion simple but also can provide us conclusions which are very close to the actual situation.

Then we give the detailed discussion of $b_1(m, r, m_0) = \frac{q(q^{m-m_0-1}-1)}{q^{m-r}-1}$ which is the upper bound of d. From the expression of b_1 we can see that b_1 and r have the same changing tendency, while b_1 and m_0 have the opposite changing tendency.

For m, since

$$\frac{b_1(m)}{b_1(m+1)} = \frac{q(q^{m-m_0-1}-1)(q^{m-r+1}-1)}{q(q^{m-m_0}-1)(q^{m-r}-1)} = \frac{q^{2m-r-m_0}-q^{m-r+1}-q^{m-m_0-1}+1}{q^{2m-r-m_0}-q^{m-r}-q^{m-m_0}+1},$$

and that

$$\begin{aligned}&(q^{2m-r-m_0}-q^{m-r+1}-q^{m-m_0-1}+1)-(q^{2m-r-m_0}-q^{m-r}-q^{m-m_0}+1)\\ &\quad= (q^{m-r}-q^{m-r+1})-(q^{m-m_0-1}-q^{m-m_0})\\ &\quad= q^{m-r}(1-q)-q^{m-m_0-1}(1-q)\\ &\quad= (q^{m-r}-q^{m-m_0-1})(1-q) \geq 0\end{aligned}$$

always holds when $0 \le m_0 < r < m$, we obtain

$$\frac{b_1(m)}{b_1(m+1)} \ge 1.$$

Therefore b_1 increases as m decreases. Based on these analyses, we have the following theorem.

Theorem 4.2. *Let $0 \le m_0 < r < m$, b_1 as in Theorem 3.4. Then*

(i) *b_1 decreases with m_0, if m and r are fixed.*
(ii) *b_1 decreases with m, if m_0 and r are fixed.*
(iii) *b_1 increases with r, if m and m_0 are fixed.*

Considering the principal part of b_1, i.e. q^{r-m_0}, we get the same conclusions for r and m_0, but since m is not contained in the principal part of b_1, we may assert m has a little impact on the change of b_1.

The following theorem tells us the wonderful relation between b_1 and its principal part q^{r-m_0} under certain conditions.

Theorem 4.3. *Let $0 \le m_0 < r < m$ and $2r \le m + m_0$. Then by referring to the definition of b_1 in Theorem 3.4, q^{r-m_0} is the largest integer less than or equal to b_1.*

Proof. Firstly, for $0 \le m_0 < r < m$,

$$b_1 - q^{r-m_0} = \frac{q(q^{m-m_0-1}-1)}{q^{m-r}-1} - q^{r-m_0} = \frac{q^{r-m_0}-q}{q^{m-r}-1} \ge 0.$$

Secondly, since $2r \le m + m_0$, we have $\frac{q^{r-m_0}-q}{q^{m-r}-1} < 1$. Hence $0 \le b_1 - q^{r-m_0} < 1$. ■

Finally, we study z as in Theorem 3.4. For clarity, we simplify the expression of z. By Lemma 2.2, we have

$$\begin{aligned}
z &= N(r-m_0,0,0;m-m_0,0,0;2(\nu-m_0)+\delta,\Delta) - dN(r-m_0,0,0;m-m_0-1,0,0;2(\nu-m_0)+\delta,\Delta)\\
&\quad + (d-1)N(r-m_0,0,0;m-m_0-2,0,0;2(\nu-m_0)+\delta,\Delta)\\
&= \frac{\prod_{i=m-r+1}^{m-m_0}(q^i-1)}{\prod_{i=1}^{r-m_0}(q^i-1)} - d\frac{\prod_{i=m-r}^{m-m_0-1}(q^i-1)}{\prod_{i=1}^{r-m_0}(q^i-1)} + (d-1)\frac{\prod_{i=m-r-1}^{m-m_0-2}(q^i-1)}{\prod_{i=1}^{r-m_0}(q^i-1)}\\
&= \frac{\prod_{i=m-r+1}^{m-m_0-2}(q^i-1)}{\prod_{i=1}^{r-m_0}(q^i-1)}\big[(q^{m-m_0-1}-1)(q^{m-m_0}-1) - d(q^{m-m_0-1}-1)(q^{m-r}-1) + (d-1)(q^{m-r-1}-1)(q^{m-r}-1)\big]\\
&= \frac{\prod_{i=m-r+1}^{m-m_0-2}(q^i-1)}{\prod_{i=1}^{r-m_0}(q^i-1)}\big[(q^{m-m_0-1}-1)(q^{m-m_0}-q^{m-r}) - (d-1)(q^{m-r}-1)(q^{m-m_0-1}-q^{m-r-1})\big]\\
&= \frac{\prod_{i=m-r+1}^{m-m_0-2}(q^i-1)}{\prod_{i=1}^{r-m_0-1}(q^i-1)}q^{m-r-1}\big[q(q^{m-m_0-1}-1) - (d-1)(q^{m-r}-1)\big].
\end{aligned}$$

Motivated by the observation after Theorems 4.1 and 4.2, we started with analyzing the principal part of z, i.e. $q^{(r-m_0)(m-r)}$. From it we get a general idea about the changing law of z with its variables that z increases as m increases, and it decreases as m_0 increases. If $r = \langle\frac{m+m_0}{2}\rangle$, z is closest to its maximum. To check our conclusions, we give a detailed analysis on z. By the above simplified expression of z, we have

$$\frac{z(m)}{z(m+1)} = \frac{\frac{\prod_{i=m-r+1}^{m-m_0-2}(q^i-1)}{\prod_{i=1}^{r-m_0-1}(q^i-1)}q^{m-r-1}\big[q(q^{m-m_0-1}-1)-(d-1)(q^{m-r}-1)\big]}{\frac{\prod_{i=m-r+2}^{m-m_0-1}(q^i-1)}{\prod_{i=1}^{r-m_0-1}(q^i-1)}q^{m-r}\big[q(q^{m-m_0}-1)-(d-1)(q^{m-r+1}-1)\big]}$$

$$= \frac{(q^{m-r+1}-1)\left[q(q^{m-m_0-1}-1)-(d-1)(q^{m-r}-1)\right]}{q(q^{m-m_0-1}-1)\left[q(q^{m-m_0}-1)-(d-1)(q^{m-r+1}-1)\right]}$$
$$= \frac{(q^{m-r+1}-1)\left[(q^{m-m_0}-q)-(d-1)(q^{m-r}-1)\right]}{(q^{m-r+1}-\frac{1}{q^{r-m_0-2}})q^{r-m_0-1}\left[(q^{m-m_0+1}-q)-(d-1)(q^{m-r+1}-1)\right]}.$$

Obviously, $(q^{m-r+1}-1) < (q^{m-r+1}-\frac{1}{q^{r-m_0-2}})$. In addition,

$$\begin{aligned}&(q^{m-m_0}-q)-(d-1)(q^{m-r}-1)-q^{r-m_0-1}\left[(q^{m-m_0+1}-q)-(d-1)(q^{m-r+1}-1)\right]\\ &\quad=(q^{m-m_0}-q)-q^{r-m_0-1}(q^{m-m_0+1}-q)-(d-1)[(q^{m-r}-1)-q^{r-m_0-1}(q^{m-r+1}-1)].\end{aligned}$$

By noting that $(q^{m-r}-1)-q^{r-m_0-1}(q^{m-r+1}-1) < 0$ and

$$\frac{q^{r-m_0-1}(q^{m-m_0+1}-q)-(q^{m-m_0}-q)}{q^{r-m_0-1}(q^{m-r+1}-1)-(q^{m-r}-1)} \geq q^{r-m_0},$$

we have that if

$$(d-1) < q^{r-m_0},$$

then

$$\begin{aligned}0 &< (q^{m-m_0}-q)-(d-1)(q^{m-r}-1)\\ &< q^{r-m_0-1}\left[(q^{m-m_0+1}-q)-(d-1)(q^{m-r+1}-1)\right],\end{aligned}$$

and

$$\frac{z(m)}{z(m+1)} < 1.$$

Above all, we have the following theorem.

Theorem 4.4. *Set $0 \leq m_0 < r < m < \nu$. When m_0, r, ν are fixed, if $d \leq q^{r-m_0}$, then z in Theorem 3.4 increases when m increases.*

For r, we have the following theorem.

Theorem 4.5. *Let $0 \leq m_0 < r < m \leq \nu$ and $0 < d \leq q+1$. For z in Theorem 3.4, z is unimodal and reaches its peak at $r = \lceil \frac{m+m_0}{2} \rceil$, when m_0, m are fixed.*

Proof. Set $m_0 < r_1 < r_2 < m$. Then

$$\frac{z(r_1)}{z(r_2)} = \frac{\frac{\prod_{i=m-r_1+1}^{m-m_0-2}(q^i-1)}{\prod_{i=1}^{r_1-m_0-1}(q^i-1)}q^{m-r_1-1}\left[q(q^{m-m_0-1}-1)-(d-1)(q^{m-r_1}-1)\right]}{\frac{\prod_{i=m-r_2+1}^{m-m_0-2}(q^i-1)}{\prod_{i=1}^{r_2-m_0-1}(q^i-1)}q^{m-r_2-1}\left[q(q^{m-m_0-1}-1)-(d-1)(q^{m-r_2}-1)\right]}$$
$$= \frac{\prod_{i=r_1-m_0}^{r_2-m_0-1}(q^i-1)q^{r_2-r_1}\left[q(q^{m-m_0-1}-1)-(d-1)(q^{m-r_1}-1)\right]}{\prod_{i=m-r_2+1}^{m-r_1}(q^i-1)\left[q(q^{m-m_0-1}-1)-(d-1)(q^{m-r_2}-1)\right]}.$$

Since

$$\frac{q^{r_2-r_1}\prod_{i=r_1-m_0}^{r_2-m_0-1}(q^i-1)}{\prod_{i=m-r_2+1}^{m-r_1}(q^i-1)} = \frac{(q^{r_1-m_0+1}-q)(q^{r_1-m_0+2}-q)\cdots(q^{r_2-m_0}-q)}{(q^{m-r_2+1}-1)(q^{m-r_2+2}-1)\cdots(q^{m-r_1}-1)},$$

and

$$\frac{q^{r_1-m_0+1}-q}{q^{m-r_1}-1} < \frac{q^{r_1-m_0+2}-q}{q^{m-r_1-1}-1} < \cdots < \frac{q^{r_2-m_0}-q}{q^{m-r_2+1}-1},$$

we have that if $r_1 < r_2 \leq \lfloor \frac{m+m_0+1}{2} \rfloor$, i.e., $r_2 - m_0 < m - r_2 + 1$, then

$$\frac{q^{r_2-r_1} \prod_{i=r_1-m_0}^{r_2-m_0-1} (q^i - 1)}{\prod_{i=m-r_2+1}^{m-r_1} (q^i - 1)} < 1.$$

By noting,

$$q(q^{m-m_0-1} - 1) - (d-1)(q^{m-r_1} - 1) \leq q(q^{m-m_0-1} - 1) - (d-1)(q^{m-r_2} - 1),$$

we have that if $r_1 < r_2 \leq \lfloor \frac{m+m_0+1}{2} \rfloor$, then

$$\frac{z(r_1)}{z(r_2)} < 1.$$

On the other hand, for $0 < d < q + 1$, apparently

$$q^{r_2-r_1}\left[q(q^{m-m_0-1} - 1) - (d-1)(q^{m-r_1} - 1)\right] > q(q^{m-m_0-1} - 1) - (d-1)(q^{m-r_2} - 1).$$

In addition, if $\lceil \frac{m+m_0}{2} \rceil \leq r_1 < r_2$, i.e., $r_1 - m_0 > m - r_1$, we have

$$\frac{q^{r_2-m_0-1} - 1}{q^{m-r_2+1} - 1} > \frac{q^{r_2-m_0-2} - 1}{q^{m-r_2+2} - 1} > \cdots > \frac{q^{r_1-m_0} - 1}{q^{m-r_1} - 1} > 1.$$

Then, if $\lceil \frac{m+m_0}{2} \rceil \leq r_1 < r_2$ and $0 < d < q + 1$,

$$\frac{z(r_1)}{z(r_2)} > 1.$$

Noting

$$\left| \frac{m+m_0}{2} - \frac{m+m_0+1}{2} \right| < 1,$$

thus z is unimodal and reaches its peak at $r = \lceil \frac{m+m_0}{2} \rceil$. ■

Acknowledgment

The authors express their sincere gratitude to the referee for a critical reading of the manuscript and many valuable suggestions and comments.

References

[1] Ding-Zhu Du, Frank K. Hwang, Pooling Designs: Group Testing in Molecular Biology, World Scientific, Singapore, 2006.
[2] Arkadii G. D'yachkov, Frank K. Hwang, Antony J. Macula, Pavel A. Vilenkin, Chih-wen Weng, A construction of pooling designs with some happy surprises, Journal of Computational Biology 12 (2005) 1129–1136.
[3] Arkadii G. D'yachkov, Anthony J. Macula, Pavel A. Vilenkin, Nonadaptive and trivial two-stage group testing with error-correcting d^e-disjunct inclusion matrices, Boylai Society Mathematical Studies, vol. 16, Springer, pp. 71–83.
[4] Hung-Lin Fu, Frank K. Hwang, A novel use of t-packings to construct d-disjunct matrices, Discrete Applied Mathematics. 154 (2006) 1759–1762.
[5] Tayuan Huang, Chih-wen Weng, Pooling spaces and non-adaptive pooling designs, Discrete Mathematics. 282 (2004) 163–169.
[6] Frank K. Hwang, On Macula's error-correcting pool designs, Discrete Mathematics 267 (2003) 311–314.
[7] Anthony J. Macula, A simple construction of d-disjunct matrices with certain constant weights, Discrete Mathematics. 162 (1996) 311–312.
[8] Anthony J. Macula, Vyacheslav V. Rykov, Sergey Yekhanin, Trivial two-stage group testing for complexes using almost disjunct matrices, Discrete Applied Mathematics 137 (2004) 97–107.
[9] Hung Q. Ngo, Ding-Zhu Du, New constructions of non-adaptive and error-tolerance pooling designs, Discrete Mathematics. 243 (2002) 161–170.
[10] Zhe-xian Wan, Geometry of Classical Groups Over Finite Fields, Sinence Press, Beijing, 2002.
[11] Weili Wu, Yaochun Huang, Xiao Huang, Yingshu Li, On error-tolerant DNA screening, Discrete Applied Mathematics 154 (2006) 1753–1758.